Finding Transfer Function of SSR

Frequency Domain Solution

$$\frac{d}{dt}x - A\cdot x = B\cdot u$$ State Space equation

$$s\cdot I\cdot X(s) - x(0) - A\cdot X(s) = B\cdot U(s)$$ Take Laplace Transform

$$(s\cdot I - A)\cdot X(s) = x(0) + B\cdot U(s)$$ Simplify

$$X(s) = (sI - A)^{-1}\cdot x(0) + (sI - A)^{-1}\cdot B\cdot U(s)$$ Simplify to obtain State Solution

$$Y(s) = C(sI - A)^{-1}\, x(0) + C(sI - A)^{-1}\cdot B\cdot U(s)$$ Multiply by "c" to obtain the output solution

"Open Loop" Transfer Function $\frac{Y(s)}{U(s)} = H(s)$

$$\frac{Y(s)}{U(s)} = H(s) = C(sI - A)^{-1}\cdot B$$ From above output solution with x(0)=0

So given a state space representaion, we can use the above definition to determine the poles and zeros of the system from

$$H(s) = \frac{C adj(sI - A)B}{|sI - A|}$$ Since $M^{-1} = \frac{adj(M)}{|M|}$

Poles are given by

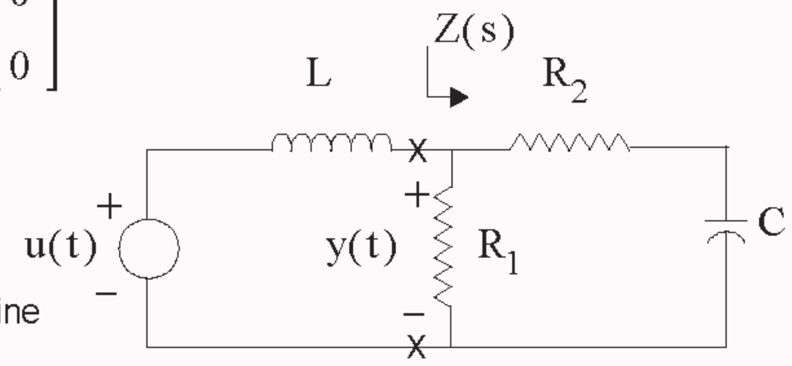
$$|sI - A| = 0$$

Zeros are given by $C\cdot adj(sI - A)\, B = 0$

Circuit Example

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u \qquad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$y = (3 \quad 0.2) \cdot x$$



Find the open loop transfer function and determine if the system is controllable and observable

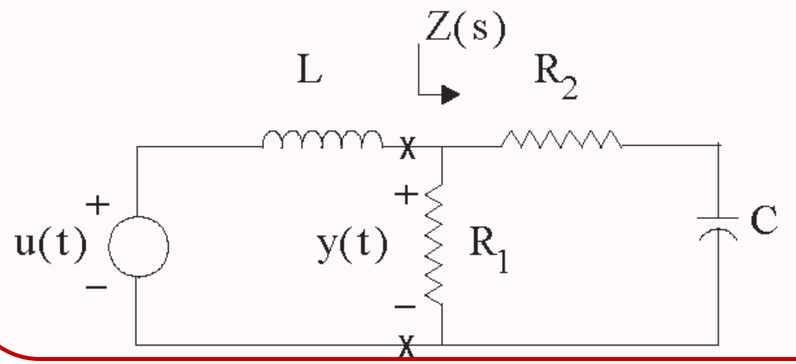
$$H(s) = C \cdot (sI - A)^{-1} \cdot B = C \cdot \begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix}^{-1} \cdot B$$ } By definition

$$H(s) = C \cdot \begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix} \cdot \frac{B}{\Delta(s)} = C \begin{bmatrix} 1 \\ s \end{bmatrix} \cdot \frac{1}{\Delta(s)} = \frac{0.2\,s + 3}{s^2 + 4\,s + 3} = \frac{0.2\,s + 3}{(s+1) \cdot (s+3)}$$ } Poles of H(s) are at -1 and -3

$$\Delta(s) = s^2 + 4\,s + 3$$ } open-loop characteristic equation

State Space Analysis: Example

Find a state-space representation for the circuit below



} Z(s) is the impedance to the right of the two x's

Note: Capacitor Impedance = $\dfrac{1}{Cs}$ Inductor Impedance = Ls

First get a transfer function for the circuit

$$\frac{Y(s)}{U(s)} = \frac{Z(s)}{Z(s) + Ls}$$ } Voltage Division

$$Z(s) = R_1 \parallel \left(R_2 + \frac{1}{Cs}\right)$$ } Calculate Z(s) from the circuit

$$Z(s) = \frac{R_1 R_2 + \frac{R_1}{Cs}}{R_1 + R_2 + \frac{1}{Cs}}$$ } Apply Parallel Law

Contd…

State Space Analysis: Example

$$Z(s) = \frac{CR_1 R_2 s + R_1}{C(R_1 + R_2) s + 1}$$ } Simplify

$$\frac{Y(s)}{U(s)} = \frac{sR_1 R_2 C + R_1}{s^2 (R_1 + R_2) LC + s(L + R_1 R_2 C) + R_1}$$ } Final Form for TF

$R_1 = 3\Omega$, $L = 3.8\ H$, $R_2 = 1\Omega$, $C = .066\ F$ } Typical Values

$$\frac{Y(s)}{U(s)} = H(s) = \frac{.2 \cdot s + 3}{s^2 + 4 \cdot s + 3}$$ } Substitute numerical values

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

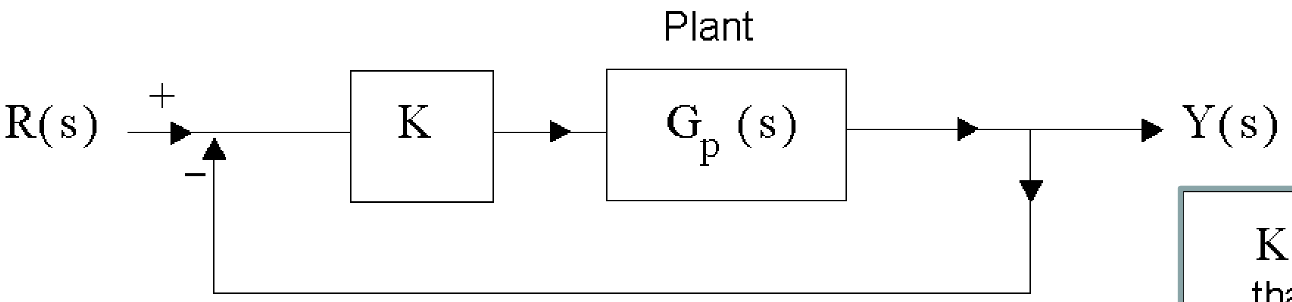
$$y = [3 \quad 0.2] x$$

} answer via RCF pattern

Previous Slide

$$Z(s) = \frac{R_1 R_2 + \frac{R_1}{Cs}}{R_1 + R_2 + \frac{1}{Cs}}$$

Root Locus: Construction and Design



K is a positive parameter that can be changed

$$\frac{Y(s)}{R(s)} = \frac{K \cdot G_p(s)}{1 + K \cdot G_p(s)}$$ } Closed Loop Transfer Function

$$G_p(s) = \frac{s^m + a_{m-1} \cdot s^{m-1} + \ldots\ldots + a_0}{s^n + b_{n-1} \cdot s^{n-1} + \ldots\ldots + b_0} = \frac{N(s)}{D(s)}$$ } General Form for the Transfer Function

Root Locus: Construction and Design (Contd.)

$N(s)$ and $D(s)$ are polynomials where $m \le n$

$$\frac{Y(s)}{R(s)} = \frac{K \cdot \frac{N(s)}{D(s)}}{1 + K \cdot \frac{N(s)}{D(s)}} = \frac{K \cdot N(s)}{D(s) + K \cdot N(s)}$$

The closed loop poles are the roots of the characteristic equation

$$\Delta(s) = D(s) + K \cdot N(s) = 0$$

The location of the roots of $\Delta(s)$ in the s-plane change as K is varied from 0 to ∞

A "Locus" of these roots plotted in the s-plane as a function of K is called the **Root Locus**

Different ways to write the same thing

$$\Delta(s) = 1 + K \cdot \frac{N(s)}{D(s)} = D(s) + K \cdot N(s) = 0$$

Gives the roots of the closed loop system

Root Locus: Rules

1. Starting points (K = 0)	The root loci start at the open-loop poles.
2. Termination points	The root loci terminate at the open-loop zeros. The open loop zeros include those at infinity.
3. Number of distinct root loci	There will be as many root loci as the largest number of finite open loop poles or zeros. For the majority of systems, the number of finite open-loop poles will be greater than the number of finite open-loop zeros.
4. Symmetry of root loci	The root loci are symmetrical with respect to the real axis.
5. Asymptote intersection	The asymptotes intersect the real axis at a point given by
centroid formula {	$\sigma_i = \dfrac{\sum open\ loop\ poles - \sum open\ loop\ zeros}{n-m}$
6. Root Locus locations on the real axis	The root loci may be found on portions of the real axis to the left of an odd number of open-loop poles and zeros.

Root Locus: Rules (Contd.)

7. Root Locus asymptotes

The root loci are asymptotic to straight lines, for large values of s, with angles given by

$$\theta = \frac{(1+2k)\pi}{n-m}$$

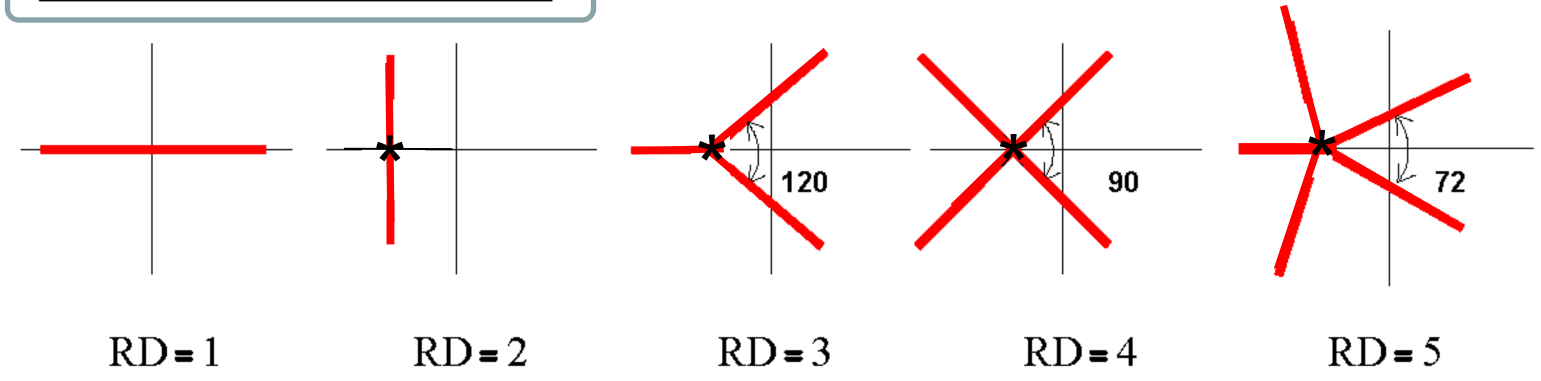
$$k = 0, 1, \ldots, n-m-1$$

$$n = \text{no. of finite open loop poles}$$

$$m = \text{no. of finite open loop zeros}$$

Let RD (relative degree) = n - m

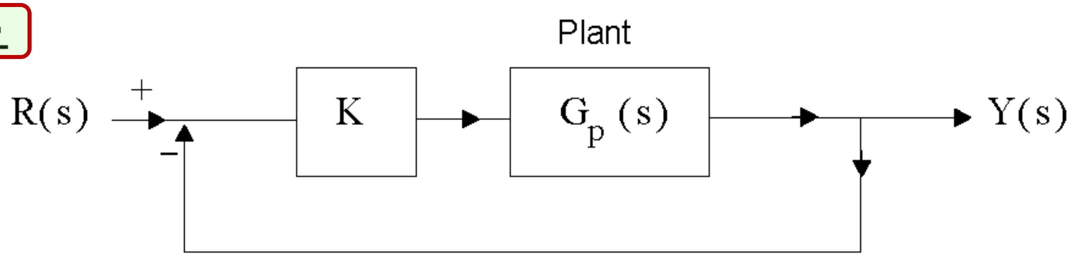
Asymptote Chart in the s plane



The centroids are marked * in the asymptote charts above

Root Locus (Contd.)

Example

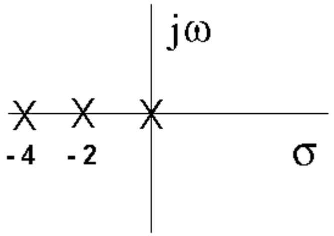


$$G_p(s) = \frac{1}{s\cdot(s+2)\cdot(s+4)}$$ } Open loop poles are at $s = 0, -2, -4$

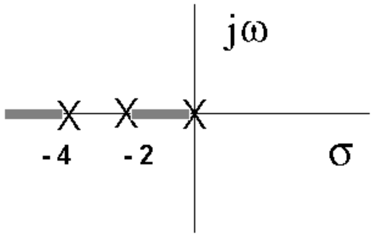
$$\frac{Y(s)}{R(s)} = \frac{K\cdot\frac{1}{s\cdot(s+2)\cdot(s+4)}}{1+K\cdot\frac{1}{s\cdot(s+2)\cdot(s+4)}}$$ } Closed Loop Transfer Function

$$\Delta(s) = 1 + K\cdot\left[\frac{1}{s\cdot(s+2)\cdot(s+4)}\right]$$ } Root Locus Form for the Denominator

Step 1. Pole Zero Plot



Step 2. Real Axis



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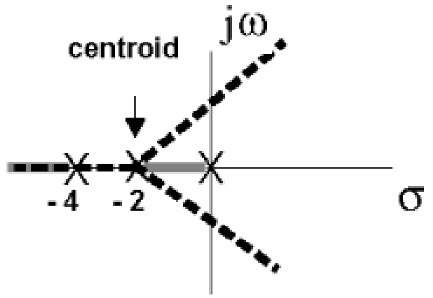
Root Locus (Contd.)

Step 3. Centroid / Asymptotes

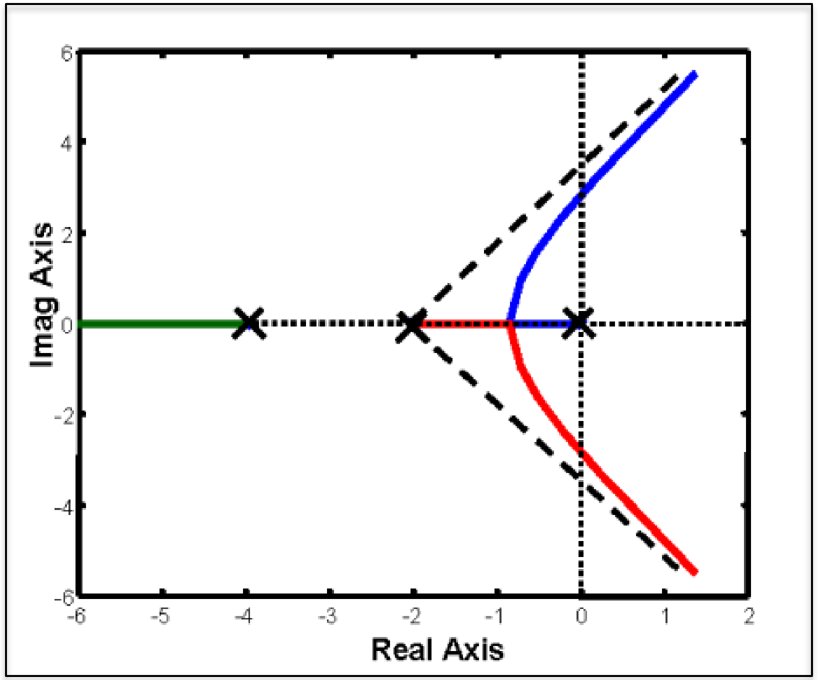
$$\text{centroid} = \frac{0 - 2 - 4 - 0}{3} = -2$$

$$RD = 3$$



Step 4. Draw the Root Locus

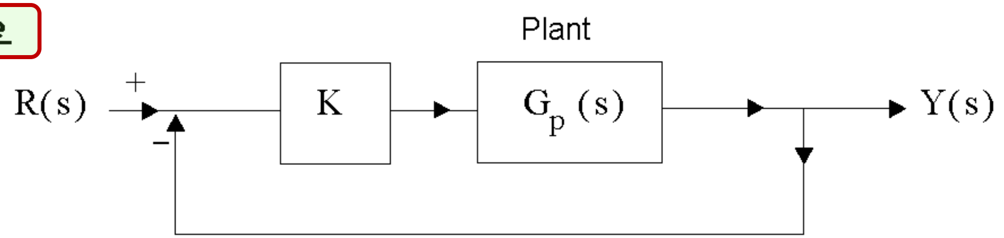
a. Locus must be symmetric about real axis
b. Open loop zeros at infinity = 3
c. Locus starts at open loop poles and goes to the open loop zeros



Note: Locus is symmetric about the real axis because complex poles always come in conjugate pairs

Root Locus (Contd.)

Example



$$G_p(s) = \frac{s+2}{(s+1)\cdot(s+3+3j)\cdot(s+3-3j)}$$

Open loop zero $s = -2$

Open loop poles $s = -1, -3 \pm j3$

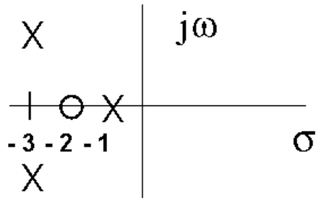
$$\frac{Y(s)}{R(s)} = \frac{K\cdot\dfrac{s+2}{(s+1)\cdot(s+3+3j)\cdot(s+3-3j)}}{1+K\cdot\dfrac{s+2}{(s+1)\cdot(s+3+3j)\cdot(s+3-3j)}}$$

Closed Loop Transfer Function

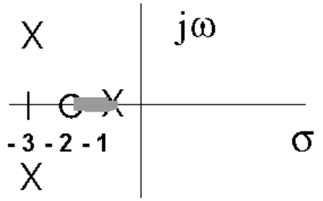
$$\Delta(s) = 1 + K\cdot\left[\frac{s+2}{(s+1)\cdot(s+3+3j)\cdot(s+3-3j)}\right]$$

Root Locus Form for the Denominator

Step 1. Pole Zero Plot



Step 2. Real Axis



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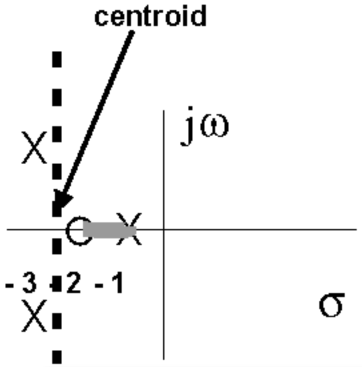
Root Locus (Contd.)

Step 3. Centroid / Asymptotes

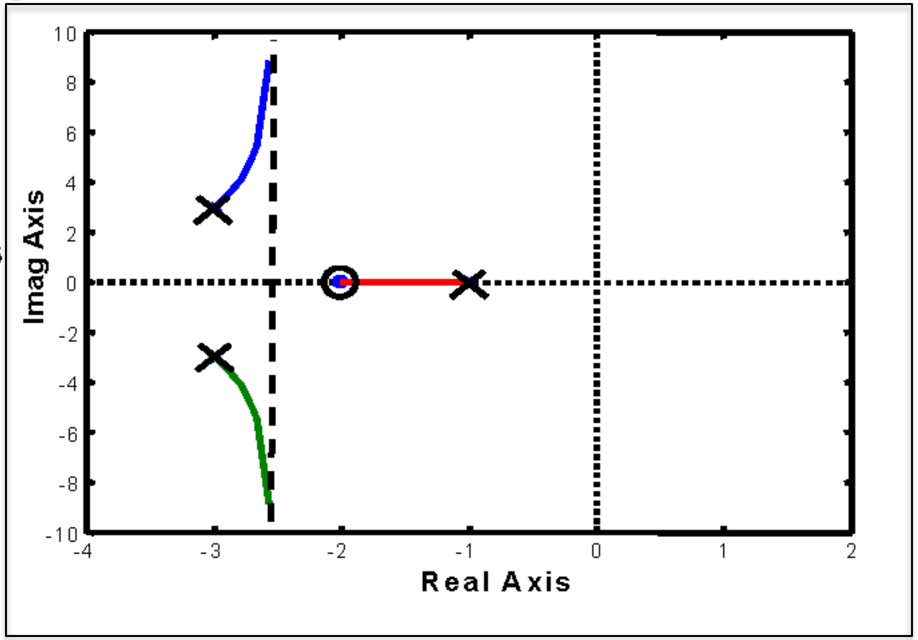
$$\text{Centroid} = \frac{-3-3-1-(-2)}{2} = \frac{-5}{2}$$

$RD = 2$



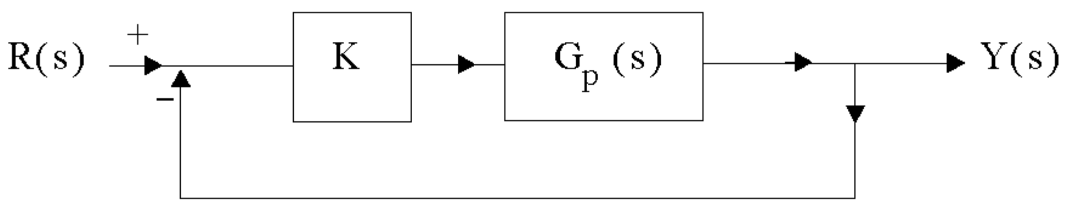
Step 4. Draw the Root Locus

a. Locus must be symmetric about real axis
b. Open loop zeros at infinity = 2
c. Locus starts at open loop poles and goes to the open loop zeros



Root Locus (Contd.)

Example

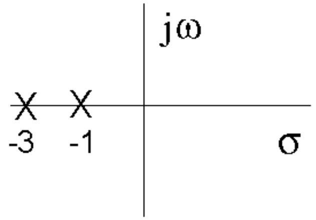


$$G_p(s) = \frac{1}{(s+1)\cdot(s+3)}$$ } Open loop poles $s = -1, -3$

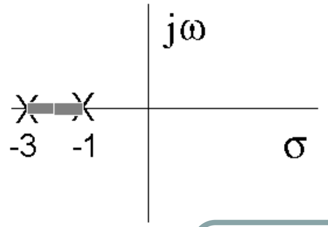
$$\frac{Y(s)}{R(s)} = \frac{K\cdot\frac{1}{(s+1)\cdot(s+3)}}{1 + K\cdot\frac{1}{(s+1)\cdot(s+3)}}$$ } Closed Loop Transfer Function

$$\Delta(s) = 1 + K\cdot\left[\frac{1}{(s+1)\cdot(s+3)}\right]$$ } Root Locus Form for the Denominator

<u>Step 1</u>. Pole Zero Plot



<u>Step 2</u>. Real Axis



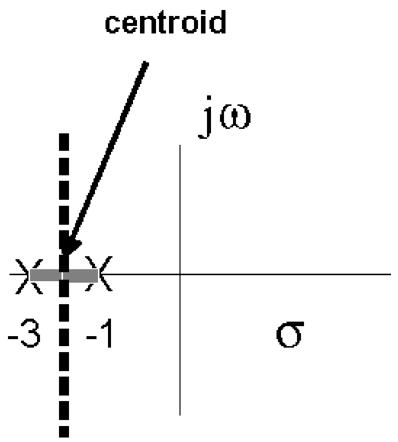
Contd…

Root Locus (Contd.)

Step 3. Centroid / Asymptotes

$$\text{Centroid} = \frac{-3 - 1}{2} = -2$$

$$RD = 2$$



Step 4. Draw the Root Locus

a. Locus must be symmetric about real axis
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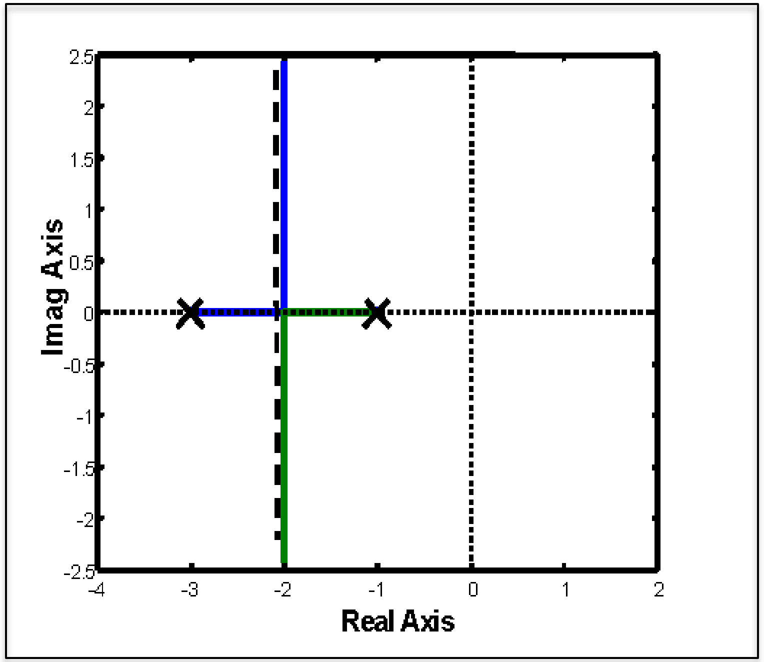
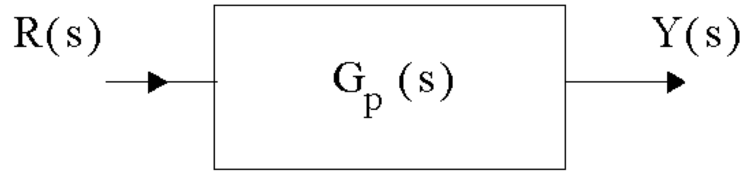
goes to the open loop zeros



Root Locus (Contd.)

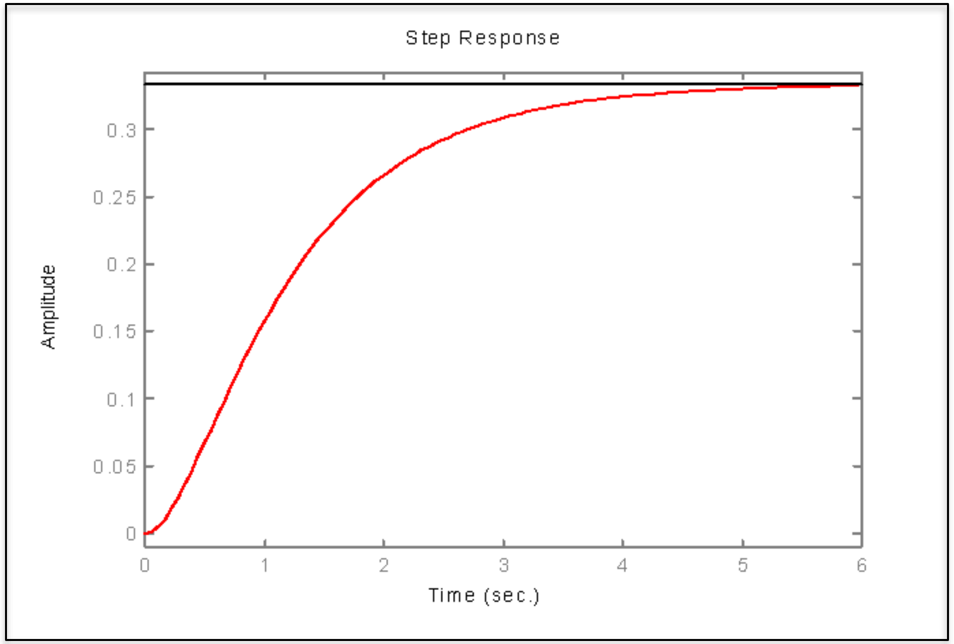
Example - Continued

Open Loop System



$$G_p(s) = \frac{1}{(s+1)\cdot(s+3)}$$

Let R(s) = Unit step



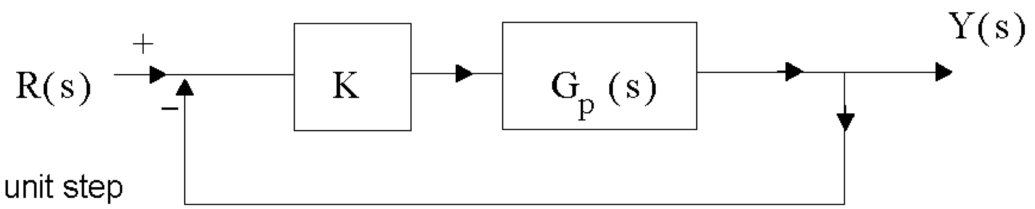
$$Y(s) = \frac{\left(\frac{1}{3}\right)}{s} + \frac{\left(\frac{-1}{2}\right)}{s+1} + \frac{\left(\frac{1}{6}\right)}{s+3}$$

$$y(t) = \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t}$$

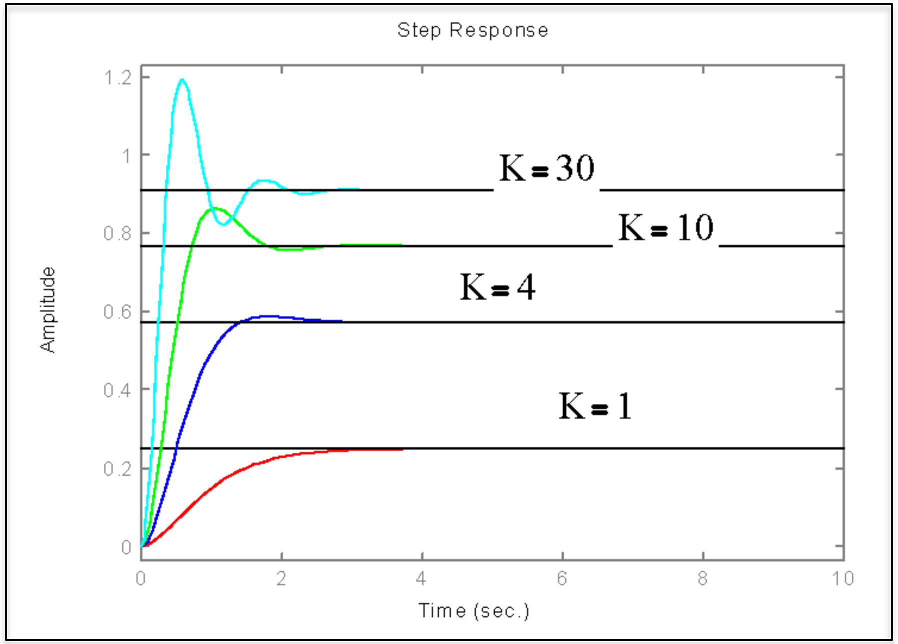
Root Locus (Contd.)

Example - Continued

Closed Loop System



Let R(s) = unit step



as K changes it is clear that the transient response can be "shaped"

Note: Higher values of K speed up the closed-loop response when compared to the open-loop response